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Excluding any graph as a minor allows a low tree-width 2-coloring

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Abstract

This article proves the conjecture of Thomas that, for every graph G, there is an integer k such that every graph with no minor isomorphic to G has a 2-coloring of either its vertices or its edges where each color induces a graph of tree-width at most k. Some generalizations are also proved.

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1. Introduction

A *vertex partition* of a graph G, into n *parts*, is a set $\{P_1, \ldots, P_n\}$ of induced subgraphs of G such that $\bigcup_{i=1}^{n} V(P_i) = V(G)$, and if $i \neq j$, then $V(P_i) \cap V(P_j) = \emptyset$. An *edge partition* of a graph G, into n *parts*, is a set $\{Q_1, \ldots, Q_n\}$ of subgraphs of G such that $\bigcup_{i=1}^{n} E(Q_i) = E(G)$, and if $i \neq j$, then $E(Q_i) \cap E(Q_j) = \emptyset$. A partition into n parts, can be associated with a coloring (of edges or vertices, as appropriate) with n colors in the obvious way.

The edge partition $\{Q_1, \ldots, Q_n\}$ of a graph G is *balanced*, *as witnessed by* a vertex partition $\{P_1, \ldots, P_n\}$ of G, if $G|P_i \subseteq G|Q_i$ for all i, and $G|(P_i \cup P_j) \subseteq G|(Q_i \cup Q_j)$, for all i, j. This is a technical condition needed later. In terms of colorings, with a set of colors C, the edge coloring, $c_E : E \to C$ is balanced, as witnessed by a vertex coloring, $c_V : V \to C$, if for every edge e with endpoints u, v, it holds that $c_E(e) \in \{c_V(u), c_V(v)\}$.

Given a graph G, a T-*decomposition* of G is a pair (T, X), where T is a graph, and for each vertex t of T, there is a *bag* $X_t \subseteq V(G)$ such that $X = (X_t : t \in V(T))$, and the following are satisfied.

(1) $\bigcup_{t \in V(T)} X_t = V(G)$.
(2) For every edge xy of G, there is a $t \in V(T)$ such that $\{x, y\} \subseteq X_t$.
(3) For every $x \in V(G)$, the subgraph of T induced by $\{t \in V(T) : x \in X_t\}$ is connected.

The *width* of (T, X) is $\max\{|X_t| - 1 : X_t \in X\}$.

If T is a tree, then (T, X) is a *tree-decomposition*. The *tree-width* of a graph G, denoted $tw(G)$, is the smallest integer w such that G has a tree-decomposition of width w. A graph is a *partial k-tree* if it has tree-width at most k. Tree-width is important not only for its theoretical application in the graph minors project, but also for its algorithmic qualities: many problems which are NP-hard for the class of all graphs are solvable in linear time for the class of graphs of tree-width at most k for every fixed k.

Given graphs G and H, G is a *minor* of H, denoted $G \leqslant_m H$, if G can be obtained from a subgraph of H by contracting edges. If a G is not a minor of H, then H is a graph *with no G-minor*. An important result of Robertson and Seymour [6] (see also [9]) is that if P is a planar graph, then there is an integer k such that every graph G with no P-minor has tree-width at most k. This is not true for any non-planar graph, as the $n \times n$ planar grid has tree-width n (see [5]).

Ding et al. [3], generalizing a relaxation of a conjecture in [1], showed that, given any surface, although a graph G embedded on that surface may have arbitrarily large tree-width, G has both a vertex partition and an edge partition into two graphs whose tree-width is bounded by a function depending only on the Euler characteristic of the surface. The set S of all graphs embedded on a surface is an example of a *minor-closed* class, that is, if $G \in S$, and $H \leqslant_m G$, then $H \in S$. Thomas [10] conjectured that every minor-closed class of graphs, other than the class of all graphs, should have this partition property. In other words, Thomas conjectured, and we prove, the following theorem:

Theorem 1.1. *For every graph K, there are integers $k_V = k_V(K)$ and $k_E = k_E(K)$, such that every graph with no K-minor has a vertex partition into two graphs with tree-width at most k_V, and an edge partition into two graphs with tree-width at most k_E.*

This follows from the following stronger theorem.

Theorem 1.2. *For every graph K and integer $j \geqslant 1$, there are integers $k_V = k_V(K,j)$ and $k_E = k_E(K,j)$, such that every graph with no K-minor has a vertex partition into $j+1$ graphs such that any j parts form a graph with tree-width at most k_V, and an edge partition into $j+1$ graphs such that any j parts form a graph with tree-width at most k_E.*

Also we prove this.

Theorem 1.3. *For every graph K and integer $j \geqslant 1$, there are integers $i_V = i_V(K,j)$ and $i_E = i_E(K,j)$, such that every graph with no K-minor has a vertex partition into i_V graphs such that any $j' \leqslant j$ parts form a graph with tree-width at most $j' - 1$, and an edge partition into i_E graphs such that any $j' \leqslant j$ parts form a graph with tree-width at most j'.*

The proofs of these theorems are completed in Section 6. These proofs will be accomplished as follows. First, an important result of Robertson and Seymour, Theorem 2.1, on the structure of graphs without a particular graph as a minor will be stated in Section 2. A corollary of Theorem 2.1, namely Corollary 2.2, will be the structural graph-theoretic result we actually use. This structure has elements both of surfaces and of tree-width in it. The main structure is a surface-like structure called an *outgrowth*. The partitions to be defined will break an outgrowth up into pieces called *layers* of bounded height. These partitions and layers are considered in Section 3. A bound on the tree-width of layers is established in Section 4. Then Section 5 deals with two other graph constructions mentioned in Section 2, namely *extensions* and *clique-joins*. Finally, as already noted, the main proofs are completed in Section 6.

To assist in the proofs, some important relationships between tree-width and graph minors are needed. The proofs of these results follow easily from the definitions of tree-width, minor, and the following. A *clique* is a complete subgraph. Given two graphs G and H, a graph J is a *clique-join* of G and H if $G \cap H$ is a clique, and J is a subgraph of $G \cup H$.

Lemma 1.4.

(1) *If $G \leqslant_m H$, then $tw(G) \leqslant tw(H)$.*
(2) *If H is a clique-join of G_1 and G_2, then $tw(H) \leqslant \max\{tw(G_1), tw(G_2)\}$.*
(3) *If there is a vertex x such that $G = H - x$, then $tw(H) \leqslant tw(G) + 1$.*

2. Excluding a graph

Let P_n be the path on vertices (in order) $t_1, \dots, t_n$. Let C_n be the circuit on vertices (in cyclic order) $t_1, \dots, t_n$.

Given a positive integer r, an *r-ring with perimeter* $t_1, \dots, t_n$ is a graph R on the vertex set $\{t_1, \dots, t_n\}$ such that there is a family of *bags* $X = (X_{t_i} : i = 1, \dots, n)$ for which:

(1) (P_n, X) is a P_n-decomposition of R of width $r-1$,
(2) for $1 \leqslant i \leqslant n$, $t_i \in X_{t_i}$.

Note that our definition of r-ring differs from that in [7] in that we require all vertices to be on the perimeter. However, if vertices were not required to be on the perimeter, an easy construction would put them on the perimeter without violating any condition in this paper.

We introduce a closely related concept. An *r-round with perimeter* $t_1, \dots, t_n$ is a graph R on the vertex set $\{t_1, \dots, t_n\}$ such that there is a family of *bags* $X = (X_{t_i} : i = 1, \dots, n)$ for which:

(1) (C_n, X) is a C_n-decomposition of R of width $r-1$,
(2) for $1 \leqslant i \leqslant n$, $t_i \in X_{t_i}$.

Observe that if $X = (X_{t_i} : i = 1, \dots, n)$ is an r-ring with perimeter $t_1, \dots, t_n$, then X is also an r-round with the same perimeter. Note that width $r-1$ means that $|X_{t_i}| \leqslant r$ for $1 \leqslant i \leqslant n$.

Let $V^* = \bigcup_{i=0}^{\infty} V^i$ denote the set of strings of letters from an alphabet V. A *supergraph* H consists of a vertex set $V = V(H)$, an edge set $E = E(H)$ and an *incidence function* $I_H : E \to V^*$. (Often we identify e with its vertex string $I_H(e)$.) For comparison, an edge in a hypergraph is (associated with) a set of vertices, whereas an edge in a supergraph is associated with an ordered list of vertices (possibly with repetition). An edge $e = v_1, \dots, v_n$ has *order* $||e|| = n$ and *size* $|e| = |\{v_1, \dots, v_n\}|$. If $||e|| = |e|$ then e is *non-degenerate* otherwise it is *degenerate*.

Two strings in V^* are *cyclically equivalent* if one can be obtained from the other by a sequence of moves that are *reversals* $v_1v_2, \dots, v_n \mapsto v_n, \dots, v_2v_1$ and *cyclings* $v_1v_2, \dots, v_n \mapsto v_2, \dots, v_nv_1$.

Throughout this paper, every edge in a supergraph has size at least two. An edge in a supergraph H is a *2–edge* if it has order two, otherwise it is a *superedge*. Let $E_2(H)$ denote the set of all 2-edges and let $E_3(H)$ denote the set of all superedges.

A supergraph H is embedded on a surface Σ as follows. Each vertex is a point. Each 2–edge, e, is the image of a continuous bijective map $f_e : [0,1] \to \Sigma$, where $f_e(0)$ and $f_e(1)$ are the endvertices of e. Each superedge, $\eta = v_1, \dots, v_n$, is the image of a continuous map $f_\eta : D_\eta \to \Sigma$ where D_η is a closed disk with distinct points, $t_1, \dots, t_n$ in cyclic order on the boundary, such that $f_\eta(t_i) = v_i$. The continuous mapping f_η is injective on $D - \{t_1, \dots, t_n\}$. Thus, if η is non-degenerate, then the image $f_\eta(D_\eta)$ is a closed disk, otherwise it is obtained from a closed disk by identifying certain points on the boundary. There are no intersections between pointsets of vertices and edges except those mentioned here.

Sometimes we identify the vertices and edges of (super)graph H with their pointsets on the surface Σ. The *faces* of H on Σ are the connected components of $\Sigma - H$.

A *crosscap* (respectively, *handle*) *reduction* of a surface is performed by replacing a crosscap (respectively, handle) by a disk. We say surface Σ' is a reduction of surface Σ, and we write $\Sigma' \leqslant \Sigma$, if Σ' is obtained from Σ by a (possibly empty) sequence of crosscap and handle reductions. We declare that an embedding of a supergraph H on Σ' is also an embedding on Σ, whenever $\Sigma' \leqslant \Sigma$.

We define crosscap and handle reductions on faces similarly. Thus, without loss of generality, we may always perform all possible crosscap and handle reductions on faces. We define two other constructions we may be able to use under certain circumstances. If a face is a surface punctured k times (that is, it has k disjoint boundaries), we *cap* the face by replacing it with k disks. We *span* the face by adding $k-1$ new 2-edges in the face, so the boundaries are joined into one boundary.

Thus, provided we can do face reductions together with cappings or spannings, we may assume that every face is an open disk.

Throughout this paper, unless otherwise noted, we assume that every graph G, every supergraph H, and every surface Σ is connected, and that every face is simply connected, that is, every face is an open disk.

Given a surface Σ and a positive integer r, let a (Σ, r)-*outgrowth* be a pair (G, H) where G is a graph, and H is a supergraph embedded on Σ, where for each superedge $\eta \in E_3(H)$, with $\eta = v_1, \ldots, v_n$, there is an r-ring R_η with perimeter $t_{1\eta}, \ldots, t_{n\eta}$, such that G is the graph obtained from H by *replacing* each superedge η by r-ring R_η, as follows. Starting with vertex disjoint copies of the graphs $H|E_2(H)$ and R_η, for $\eta \in E_3(H)$, identify vertex $t_{i\eta}$ with $f_\eta(t_{i\eta})$ for every $\eta \in E_3(H)$ and every $i \leqslant ||\eta||$. Clearly, $V(G) = V(H)$.

Given a surface Σ and positive integers r, s, let a $(\Sigma, (r, s))$-*outgrowth* be a triple (G, H, η_0) where G is a graph, H is a supergraph embedded on Σ, and $\eta_0 \in E_3(H)$, such that G is the graph obtained from H by replacing η_0 by an s-round and by replacing every other superedge by an r-ring, as described in detail above. Again, clearly $V(G) = V(H)$.

If (G, H) or (G, H, η_0) is an outgrowth, we refer to the graph G as the graph *from* the outgrowth. Note that in [7], the graph itself is referred to as the outgrowth. Note also that our definition of outgrowth differs from that in [7] in that we allow superedges to be degenerate.

If Θ is a non-empty set of surfaces, then (G, H) is a (Θ, r)-*outgrowth* if there is a $\Sigma \in \Theta$ such that (G, H) is a (Σ, r)-outgrowth. Let K_0 be the empty graph. It is intended that (K_0, K_0) is a (Σ, r)-outgrowth for all possible values of Σ and r. Thus for any set Θ of surfaces, (K_0, K_0) is a (Θ, r)-outgrowth. If Θ is empty, then let (K_0, K_0) be the only (Θ, r)-outgrowth. We repeat all these definitions where r is replaced by (r, s).

Given an integer k, let G' be a $(\leqslant k)$-*vertex extension* of G if there is a set S of at most k vertices of H such that $G = G' - S$. For example, the set of apex graphs is the set of graphs which are $(\leqslant 1)$-vertex extensions of planar graphs.

The following is a major structural result from [7]. This theorem followed from an important step in proving the Graph Minors Theorem [8].

Theorem 2.1 (Robertson and Seymour). *Let K be a graph, and let $\Theta(K)$ be the set of all surfaces in which K cannot be embedded. Then there are numbers $r(K)$, $w(K)$ and $d(K)$ such that every graph with no K-minor may be constructed by clique-joins, starting from $(\leqslant w(K))$-vertex extensions of graphs from $(\Theta(K), r(K))$-outgrowths (G_1, H_1), where $|E_3(H_1)| \leqslant d(K)$, and H_1 has no degenerate edges.*

It should be mentioned that Theorem 2.1 was proved with a slightly different understanding of r-rings as noted above. However, it is easy to show that this theorem remains valid if we additionally require that all vertices of an r-ring are on the perimeter, as we do in this paper.

We wish to use a modified version of the structural theorem above, by removing the bound $d(K)$ on the number of superedges (and r-rings), and by removing the requirement that superedges be non-degenerate.

Corollary 2.2. *Let K be a graph, and let $\Theta(K)$ be the set of all surfaces in which K cannot be embedded. Then there are numbers $r(K)$ and $w(K)$ such that every graph with no K-minor may be constructed by clique-joins, starting from $(\leqslant w(K))$-vertex extensions of graphs from $(\Theta(K), r(K))$-outgrowths.*

Note that if K is a planar graph, then $\Theta(K) = \emptyset$, and a graph with no K-minor may be constructed by clique-joins, starting from graphs on at most $w(K)$ vertices. This special case of Theorem 2.1 (or Corollary 2.2) that, for every planar graph P, a graph with no P-minor has low tree-width, appears in [6] (see also [9]).

3. Layers

The main step towards proving Theorem 1.2 is finding an edge and a vertex $(j+1)$-coloring, of a (Σ, r)-outgrowth (G, H), such that any j colors form a graph with bounded tree-width. (The bound would involve Σ, r and j.) Clearly there will be no loss of generality in adding new edges to G and H, nor in deleting parallel 2-edges. We will describe two such constructions shortly.

We first examine the boundary of a face, which may be more complicated than just a circuit. For a face F, let $\bar{F}$ denote its closure and let $\partial F = \bar{F} - F$ denote its boundary. Let D_F be a closed disk, with interior D'_F and boundary ∂D_F. A continuous injective map from D'_F to F extends uniquely to a continuous map $f_F : D_F \to \bar{F}$. Let $t_1, \dots, t_n$ be the list, in cyclic order, of all points of ∂D_F that are mapped to vertices. Let $v_i = f_F(t_i)$ for $i = 1, \ldots, n$. The string $v_1, \ldots, v_n$ is a *vertex-boundary of F*. This string is unique up to cyclic equivalence.

The face F has *order* $||F|| = n$ and *size* $|F| = |\{v_1, \ldots, v_n\}|$. If $||e|| = |e|$, then F is *non-degenerate* otherwise it is *degenerate*.

For a face F with vertex-boundary $v_1, \ldots, v_n$, $n \geqslant 3$, we *fill F with a trivial superedge* η_F *and r-ring* R_{η_F} by adding a new superedge $\eta_F = v_1, \ldots, v_n$ embedded in the natural way. Note that the pointset of F is replaced by the pointset of η_F (minus its vertices) and n new faces of order two incident with η_F. The *trivial* r-ring R_{η_F} has no edges and has bag $X_t = \{t\}$ for each perimeter vertex t. Similarly, we may fill a face with a trivial s-round.

For any superedge $\eta = v_1, \ldots, v_n$ we *encycle* η by adding those 2-edges $v_i v_{i+1}$ (reading indices modulo n) to G and H which are not already present and embedded in the natural way. The new edges are also embedded in the natural way.

Suppose F_0 is a face of supergraph H. We say that H is *full* (respectively, (H, F_0) is *full*) if every face (respectively, other than F_0) has order two, and is incident with one 2-edge and one superedge. We *fill H* (respectively, (H, F_0)) by performing all possible face fillings (respectively, except on F_0) and all possible encyclings.

The set of *elements* of a supergraph H on surface Σ is the set $P(H)$ of all vertices, edges and faces of H on Σ.

We define a (symmetric) incidence relation on the elements of H. A vertex v is *incident* with edge $e = v_1, \ldots, v_n$ if $v \in \{v_1, \ldots, v_n\}$. A vertex v is *incident* with face F if $v \in \bar{F}$. A face F is *incident* with an edge e if $\bar{F}$ intersects the pointset of e in a point other than a vertex. No other incidences are possible; in particular, elements of the same type cannot be incident.

We define a metric, $d : P(H) \times P(H) \to \mathbb{R}$ on the elements of H. Let $d(a, b) = \frac{m}{2}$, where m is the least integer such that $a = p_0$ and $b = p_m$ and consecutive entries in the sequence $p_0, \ldots, p_m$ are incident. In particular $d(a, b) = \frac{1}{2}$ if and only if a and b are incident. Moreover, vertices incident with a common face (or edge) are at distance 1.

Given a connected supergraph H drawn on a surface Σ and a face F_0, the *height*, $h^H_{F_0}(x) = h(x)$ of an element $x \in P(H)$, *relative to* F_0, is given by $h(x) = d(F_0, x) - \frac{1}{2}$. Observe that, if (H, F_0) is full, then every vertex has integer height.

If (G, H) is a (Σ, r)-outgrowth, F_0 is a face of H and $2h \geqslant 0$ is an integer, then (G, H, F_0) is a (Σ, r, h)-*layer* provided that every vertex and edge of H has height at most h. Layers are of interest because the vertex and edge partitions of (Σ, r)-outgrowths that we will define have components which are, respectively (Σ, r, h)-layers for some h to be specified later. We will prove this assertion after some definitions. (The following section proves that these layers have bounded tree-width.)

We use a traditional approach to defining a vertex and edge partition of an outgrowth (G, H) based upon distance from a given face F_0 of the supergraph. Without loss of generality (H, F_0) is full. For each integer $h \geqslant 0$ let V_h be the set of vertices at height h. For integers $0 \leqslant i \leqslant j$, let $E_{i,j}(G)$ be the set of edges of G which have one end at height i and the other at height j. (Note that $E_{i,j}(G) = \emptyset$ unless $i - j \leqslant 1$.) Define $E_{i,j}(H)$ similarly. Observe that the edges in $E_{h-1,h}$ all have height

$h-\frac{1}{2}$, whereas the edges in $E_{h,h}$ may have heights $h-\frac{1}{2}$, h or $h+\frac{1}{2}$, and if $h(e)=h$ then e is a 2-edge.

It is convenient to define vertex and edge partitions via colorings. For an integer $l \geqslant 2$, the set of colors will be $\mathbb{Z}_l$, the *integers* mod l. The *canonical vertex l-coloring* assigns color $h(\bmod\ l)$ to all vertices in V_h. The *canonical edge l-coloring* assigns color $h(\bmod\ l)$ to all edges in $E_{h-1,h} \cup E_{h,h}$. Note that this edge-coloring is balanced, as witnessed by the given vertex coloring.

We will be interested in subgraphs formed by some $1 \leqslant j < l$ of these colors, and in bounding the tree-width of (the components of) these subgraphs. Since adjacent vertices or edges can differ in height by at most one, it is sufficient to bound the tree-width of the following two types. Here $0 \leqslant a \leqslant b = a+j-1$.

$$G^V_{[a,b]} = G|\left(\bigcup_{h=a}^{b} V_h\right),$$

$$G^E_{[a,b]} = G|\left(\bigcup_{h=a}^{b} (E_{h-1,h} \cup E_{h,h})\right).$$

Let $E'_a = \{e \in E_{a,a} : h(e) \geqslant a\}$. Let

$$\hat{G}_{[a,b]} = G|\left[\left(\bigcup_{h=a}^{b} (E_{h-1,h} \cup E_{h,h})\right) \cup E'_a\right].$$

We wish to express $\hat{G}_{[a,b]}$ as an outgrowth of an appropriate supergraph. This is done naturally, but we provide details. Suppose, in general, $G' = (V', E')$ is a subgraph of G where (G,H) is a (Σ, r)-outgrowth with F_0 a face of H, and r-ring R_η associated with each superedge $\eta \in E_3(H)$. We define an outgrowth (G', H') and a layer (G', H', F'_0) as follows. Let $V(H') = V(G')$. Delete every edge $e \in E - E'$ from either $E_2(H)$ or some r-ring R_η. Also, delete the edges in $(E-E') \cap E_2(H)$ from the drawing of H on Σ. After deleting these edges, this leaves the vertices to be deleted, $V - V'$, isolated. Remove each vertex in $V - V'$ from the drawing of H, from the vertex list of each superedge, from every r-ring perimeter and every bag. If, after these deletions, the superedge η has size less than two, then delete it. If, after these deletions, the superedge η has size two, say with incident vertices u, v, and edge uv is in R_η, then replace η with a 2-edge uv, otherwise just delete it. Otherwise, make the new embedding of disk D_η, representing η, a subset of the old embedding such that it only hits the appropriate remaining vertices. Let F'_0 be the face of H' that contains the pointset of F_0. Note that F'_0 need not be simply connected.

Define $(\hat{G}_{[a,b]}, \hat{H}_{[a,b]}, \hat{F}_{[a,b]})$ similarly to (G', H', F'_0).

Continue by reducing, capping and filling all faces other than $\hat{F}_{[a,b]}$. Reduce and span face $\hat{F}_{[a,b]}$, to obtain $(G_{[a,b]}, H_{[a,b]}, F_{[a,b]})$ which is full, by construction.

Lemma 3.1. *Suppose (G, H) is a (Σ, r)-outgrowth, with face F_0, such that (H, F_0) is full. For integers $l \geqslant 2$, $1 \leqslant j < l$, $0 \leqslant a \leqslant b = a + j - 1$, $(G_{[a,b]}, H_{[a,b]}, F_{[a,b]})$ is a full $(\Sigma, r, j + \frac{1}{2})$-layer. Moreover $G^V_{[a,b]}$ and $G^E_{[a,b]}$ are subgraphs of $G_{[a,b]}$.*

Proof. These statements are clear except for the bound $j + \frac{1}{2}$ on height of $L = (G_{[a,b]}, H_{[a,b]}, F_{[a,b]})$. Note that the edges added by filling faces other than $\hat{F}_{[a,b]}$ all have endpoints in $V_b(G)$ and without loss of generality we may assume these edges were already in G. If $a = 0$, the result is immediate, so we may suppose $a > 0$. Now, considering heights in the original (G, H, F_0), the vertices of $G_{[a,b]}$ have heights ranging from $a - 1$ to b and the edges of $G_{[a,b]}$ have heights ranging from $a - \frac{1}{2}$ to $b + \frac{1}{2}$. Now all the vertices and edges of $G_{[a,b]}$ that had height $a - 1$ in (G, H, F_0), are incident with the face $F_{[a,b]}$ in $H_{[a,b]}$, so their height in L is zero. Consequently, all other vertices and edges of $G^E_{[a,b]}$ have height in L that is $a - 1$ less than its original height in (G, H, F_0). Thus L has height at most $b + \frac{1}{2} - (a - 1) = j + \frac{1}{2}$ as required. $\square$

The above bounds on height are best possible.

4. Bounding the tree-width of (Σ, r, h)-layers

This section establishes a bound on the tree-width of a (Σ, r, h)-layer in terms of Σ, r and h.

A $(\Sigma, (r, s), h)$-layer is a triple $(G, H, \eta(F))$ where H has a special superedge $\eta(F)$ and associated s-round $R_{\eta(F)}$; every other superedge has an associated r-ring; G is obtained from H by replacing superedges by the s-round and r-rings; $(G - E(R_{\eta(F)}), H - \eta(F), F)$ is a (Σ, r, h)-layer, where F is created by the deletion of $\eta(F)$. (Note that F is not a face of H.) The height of $\eta(F)$, and its incident faces, is declared to be $-\frac{1}{2}$. All other vertices and edges have the same height in $(G, H, \eta(F))$ as in $(G - E(R_{\eta(F)}), H - \eta(F), F)$.

For a surface Σ, let $\chi(\Sigma)$ denote the Euler characteristic of Σ.

Let $\langle h \rangle$ denote $h - \lfloor h \rfloor$, the fractional part of h. The following is the main theorem of this section.

Theorem 4.1. *If $(G, H, \eta(F))$ is a full $(\Sigma, (r, s), h)$-layer, then G has tree-width at most*

$$\begin{aligned}&(2 - \chi(\Sigma))[2\lfloor h \rfloor(2r + 1) + 2\langle h \rangle(r + 1) + 2s - 1] \\ &\quad + [3\lfloor h \rfloor(2r + 1) + 4\langle h \rangle(r + 1) + 3s - 2].\end{aligned}$$

The proof of the theorem will follow some lemmas and definitions. Before stating these, we prove two easy corollaries, which will be used in the proof of the main theorem appearing in the next section.

Corollary 4.2. *If (G,H,F) is a full (Σ,r,h)-layer, then G has tree-width at most*

$$\begin{aligned}&(2-\chi(\Sigma))[2\lfloor h\rfloor(2r+1)+2\langle h\rangle(r+1)+1]\\&\quad+[3\lfloor h\rfloor(2r+1)+4\langle h\rangle(r+1)+1].\end{aligned}$$

Proof. Fill face F with an s-round with $s=1$ and apply Theorem 4.1. $\square$

Corollary 4.3. *Let G be a connected graph such that (G,H) is (Σ,r)-outgrowth, and let F be a face of H. Then G has a balanced edge and a vertex l-coloring, such that any $j<l$ colors form a graph with tree-width at most*

$$(2-\chi(\Sigma))[2j(2r+1)+r+2]+[3j(2r+1)+2r+3].$$

Proof. This follows from Lemma 3.1, and the preceding discussion, with $h=j+\frac{1}{2}$. $\square$

The 2-*degree* of a vertex v in a supergraph H, denoted $\deg_2 v$ is the degree of v in $H|E_2(H)$. A full supergraph is *subcubic* if every vertex has 2-degree at most three.

For $L=(G,H,\eta(F))$, let $X^L_{\eta,t}$ denote the bag associated with superedge η and perimeter vertex t on R_η for L.

Note that H being full implies that the edges listed cyclically around any vertex alternate between 2-edges and superedges.

Lemma 4.4. *If there is a counterexample to Theorem* 4.1, *then there is a counterexample with H full and subcubic.*

Proof. As discussed above, we can suppose the supergraph H is full without loss of generality. The remainder of the proof is similar to the proof that any graph can be obtained from a subcubic graph by contraction, although here we must preserve height and other structure.

Suppose x is a vertex with 2-degree more than three. Let the height of x be j. Let the cyclic ordering of edges around x be $\eta_1,e_1,\dots,\eta_m,e_m$ (where each η_i is a superedge and each e_i is a 2-edge) such that, without loss of generality, the height of η_1 is $j-\frac{1}{2}$. Suppose $l\in\{3,\dots,m-1\}$.

We define the *height preserving decontraction of x through (η_1,η_l)* as follows, to obtain $L'=(G',H',\eta(F))$ from $L=(G,H,\eta(F))$. Remove x from H and add two new vertices x' and x''. Add a new 2-edge $e_x=x'x''$. Each edge in $\eta_1,e_1,\dots,e_{l-1},\eta_l$ is declared to be incident with x'. Each edge in $\eta_l,e_l,\dots,\eta_m,e_m,\eta_1$ is declared to be incident with x''. H' is embedded naturally. For $y\in\{x',x''\}$ and η incident with y in H', the bag $X^{L'}_{\eta,y}=X^L_{\eta,x}\cup\{y\}-\{x\}$. For any other bag Y, with $x\in Y$ associated with η_1 or η_l in H we change Y to $Y\cup\{x'\}-\{x\}$ or $Y\cup\{x''\}-\{x\}$ so as to preserve the required properties for r-rings and s-rounds. This creates a triple $(G',H',\eta(F))$ where $G=G'/e_x$.

Note, by the choice of η_1 this operation does not increase height. By finitely many height preserving decontractions we obtain a $(\Sigma, (r, s), h)$-layer, $(G'', H'', \eta(F))$ such that H'' is subcubic and G'' is a minor of G. By Lemma 1.4 (1) the result follows. $\square$

Sometimes it is useful to rotate the perimeter around an r-ring. The next simple lemma shows that this may be done by adding a factor of two to its width.

Lemma 4.5. *If R is an r-ring with perimeter $t_1, \ldots, t_n$, then for each i such that $2 \leqslant i \leqslant n$, R is a $(2r)$-ring with perimeter $t_i, \ldots, t_n, t_1, \ldots, t_{i-1}$.*

Proof. Let R be an r-ring with perimeter $t_1, \ldots, t_n$ by means of bags $X_1, \ldots, X_n$. Let $i \in \{2, \ldots, n\}$ be given. For each $j \in \{1, \ldots, n\}$, let $Y_j := X_i \cup X_j$. Then the bags $Y_i, \ldots, Y_n, Y_1, \ldots, Y_{i-1}$ give the result. $\square$

Lemma 4.6. *If there is a counterexample to Theorem 4.1, then there is a counterexample with H full, subcubic with height $h < 1$.*

Proof. Suppose the $(\Sigma, (r, s), h)$-layer, $L = (G, H, \eta(F))$ is a counterexample where H is full and subcubic, with the height h as small as possible. Suppose $h \geqslant 1$. We shall construct a $(\Sigma, (r, s + 2r + 1), h - 1)$-layer, $L' = (G, H', \eta(F'))$, as follows, to obtain a contradiction. For $i \in \{2, 3\}$ let $E_i^h(H)$ denote the set of edges in $E_i(H)$ which have height h. Since H is full and subcubic, for every superedge $\eta \in E_3^{1/2}(H)$, there is a 2-edge $e(\eta) \in E_2^0(H)$ such that η and $e(\eta)$ bound a face. By finitely many height preserving decontractions we may ensure that these 2-edges form a matching. Applying Lemma 4.5 we rotate the r-ring R_η associated with each $\eta \in E_3^{1/2}(H)$, to obtain a $2r$-ring R'_η with perimeter $t_1, \ldots, t_n$ and bags $X_1, \ldots, X_n$, say, where $e(\eta) = t_1 t_n$. Construct H' from H by deleting $\eta(F)$, and every η and $e(\eta)$ for $\eta \in E_3^{1/2}(H)$, creating a face F', and then filling F' with a superedge $\eta(F')$.

We now need to associate with $\eta(F')$ an $(s + 2r + 1)$-round, by specifying all the bags, so that $(G, H', \eta(F'))$ has the required properties. For each $\eta \in E_3^{1/2}(H)$, with R'_η, $t_1, \ldots, t_n$ and $X_{t_1}, \ldots, X_{t_n}$ as above, let $X^{L'}_{\eta(F'), t_i} = X^{L}_{\eta(F), t_1} \cup X_{t_i}$ for $i \in \{1, \ldots, n-1\}$ and let $X^{L'}_{\eta(F'), t_n} = X^{L}_{\eta(F), t_n} \cup X_{t_n} \cup \{t_1\}$. Any other vertex t of $\eta(F')$ corresponds naturally to a vertex of $\eta(F)$ so we leave the bag unchanged.

It is routine to check that L' has the required properties and the lemma follows. $\square$

We now complete the proof of Theorem 4.1.

Proof. Suppose $(G, H, \eta(F))$ is a $(\Sigma, (r, s), h)$-layer which contradicts the theorem. By the above lemmas we may assume that H is full and subcubic and that $h \in \{0, \frac{1}{2}\}$.

Suppose first that $h = 0$. Then H has a single superedge $\eta(F)$. Let $H_2 = H - \eta(F)$. Then H_2 is a graph on Σ with a single face F. By Euler's formula, there is a set C of exactly $2-\chi(\Sigma)$ edges such that $T = H_2 - C$ is a spanning tree of H_2. For each $x \in V(G)$ let T_x be the set of all vertices $v \in V(G)$ such that x is in some bag associated with v. Clearly $H_2|T_x$ is connected for every $x \in V(G)$.

Now each 2-edge is associated with four entries (each with a corresponding bag) in the vertex-list of the face F. Let Y_e and Z_e be the two bags at one end of e. For any x, if $e \in H_2|T_x$, then $x \in Y_e \cup Z_e$. Let $D = \bigcup_{e \in C}(x \in Y_e \cup Z_e)$. Let $G' = G - D$. Now G' is obtained from G by the deletion of at most $(2-\chi(\Sigma))(2s-1)$ vertices so by Lemma 1.4,

$$tw(G) \leqslant tw(G') + (2-\chi(\Sigma))(2s-1).$$

A tree-decomposition of G' of width $3s-2$ (that is, with bag size at most $3s-1$) is obtained as follows, using tree T. Arbitrarily choose a root z of T. For each $x \in V(T)$ let the bag be the union of the bags (with D removed) associated with x in L together with the predecessor (if any) of x in the rooted tree. The result follows for $h = 0$.

The result for $h = \frac{1}{2}$ follows similarly after a reduction similar to that in the proof of Lemma 4.6. We just briefly outline the modifications to the arguments for this case. We do not need to rotate any r-rings for this construction, although we may first have to do some decontractions (at no cost to the bounds). We reduce to the $h = 0$ case where, at worst, each vertex is associated with one bag of size s and two of size $s+r+1$. The edges in $E_2^{1/2}(H)$ form a matching and so we assume the tree T contains all of these to obtain the required bound. $\square$

5. Extensions and clique sums

In this section we extend the partition results from outgrowths to minor-closed classes. It will become apparent why the edge partitions of the previous sections were required to be balanced. This property is very useful in dealing with clique-joins.

For a set of surfaces Θ, and integer $r \geqslant 1$, let $\Gamma_{\Theta,r}$ be the set of all graphs from (Θ, r)-outgrowths.

For a set of graphs Γ and integer $w \geqslant 0$, let $\Gamma \oplus w$ be the set of all pairs (G, W) where G is a graph and $W \subseteq V(G)$ such that $|W| \leqslant w$ and $(G - W) \in \Gamma$. We call the vertices in W the *extra* vertices of G.

Let $\Gamma + w$ be the set of graphs G for which there exists W such that $(G, W) \in \Gamma \oplus w$, that is, $\Gamma + w$ is the set of all graphs that are $(\leqslant w)$-vertex extensions of graphs in Γ.

Define $\Gamma + w\Delta$ inductively as follows:

(1) If $G \in \Gamma + w$, then $G \in \Gamma + w\Delta$.
(2) If $G_1 \in \Gamma + w\Delta$ and $(G_2, W) \in \Gamma \oplus w$ and G is a clique-join of G_1 and G_2 on a clique Q with $V(Q) \subseteq W$, then $G \in \Gamma + w\Delta$.

The only restriction from taking arbitrary clique-joins is that $V(Q) \subseteq W$. Theorem 2.1 and Corollary 2.2 still hold with such a restriction imposed [4], and so we get a further corollary.

Corollary 5.1. *Let Γ be a minor closed class of graphs other than the class of all graphs. Then there is a finite set of surfaces, Θ, and there are numbers r and w such that*

$$\Gamma \subseteq (\Gamma_{\Theta,r}) + w\Delta.$$

Proof. Choose graph $K \notin \Gamma$. Let $\Theta = \Theta(K)$, $r = r(K)$, $w = w(K)$. The result follows from Corollary 2.2 modified with the above-noted restriction on clique-joins. $\square$

Let C be a set of colors. Let $G = (V, E)$ be a graph. For an edge coloring $c_E : E \to C$ of graph G, and $B \subseteq C$, let $E(B) = \{e \in E : c_E(e) \in B\}$. For a vertex coloring $c_V : V \to C$ of graph G, and $B \subseteq C$, let $V(B) = \{v \in V : c_V(v) \in B\}$.

Suppose $k_E^B \in \mathbb{Z} \cup \{\infty\}$ for all $B \subseteq C$. We say that graph G has a $(k_E^B : B \subseteq C)$ edge coloring $c_E : E \to C$ if $tw(G|E(B)) \leqslant k_E^B$ for all $B \subseteq C$. Make a similar definition for vertex coloring.

If the edge coloring, $c_E : E \to C$ is balanced, as witnessed by a vertex coloring, $c_V : V \to C$, then we may combine these into a *balanced full coloring* $c = (c_E \cup c_V) : (E \cup V) \to C$. Recall, this has the property that for every edge e with endpoints u, v, it holds that $c(e) \in \{c(u), c(v)\}$. We call this condition *the endpoint rule*. We say c is a *balanced $(k_E^B : B \subseteq C)$ full coloring*, if $C|E$ is a balanced $(k_E^B : B \subseteq C)$ edge coloring.

Lemma 5.2. *Suppose that Γ is a set of graphs, and $w \geqslant 0$ is an integer. Let C be a set of colors. Suppose $k_E^B \in \mathbb{Z} \cup \{\infty\}$ and $k_V^B \in \mathbb{Z} \cup \{\infty\}$, for all $B \subseteq C$. Suppose $G \in \Gamma + w\Delta$.*

(1) *If every graph in Γ has a $(k_V^B : B \subseteq C)$ vertex coloring, then G has a $(k_V^B + w : B \subseteq C)$ vertex coloring.*
(2) *If every graph in Γ has a balanced $(k_E^B : B \subseteq C)$ edge (and full) coloring, then G has a balanced $(k_E^B + w : B \subseteq C)$ edge (and full) coloring.*

Proof. (1) Let Γ be a set of graphs, such that every graph in Γ has a $(k_V^B : B \subseteq C)$ vertex coloring. Suppose $G \in \Gamma + w\Delta$.

If $G \in \Gamma + w$, then there exists W such that $(G, W) \in \Gamma \oplus w$. Thus $G - W \in \Gamma$ and so has a $(k_V^B : B \subseteq C)$ vertex coloring. If we extend the coloring from $G - W$ to G by arbitrarily coloring W, then the result follows, using Lemma 1.4(3).

Now suppose $G_1 \in \Gamma + w\Delta$ and $(G_2, W) \in \Gamma \oplus w$ and G is a clique-join of G_1 and G_2 on a clique Q with $V(Q) \subseteq W$. By induction there is a $(k_V^B + w : B \subseteq C)$ vertex coloring of G_1 and there is a $(k_V^B : B \subseteq C)$ vertex coloring of $G_2 - W$. Note that the graphs G_1 and $G_2 - W$ are vertex disjoint so we may use these two colorings to vertex color $G_1 \cup (G_2 - W)$. This leaves the vertices in $W - V(Q)$ to be colored, and we do so arbitrarily.

Observe, for any $B \subseteq C$, that $G|V(B)$ is a clique-join of $G_1|V(B)$ and $G_2|V(B)$, on the clique with vertex set $V(Q) \cap V(B)$. By Lemma 1.4(3), $tw(G_2|V(B)) \leqslant tw((G_2 - W)|V(B)) + w \leqslant k_V^B + w$. So by Lemma 1.4(2), $tw(G|V(B)) \leqslant k_V^B + w$, as required.

(2) Let Γ be a set of graphs, such that every graph in Γ has a $(k_E^B : B \subseteq C)$ edge coloring. Suppose $G \in \Gamma + w\Delta$.

We wish to find a *balanced* $(k_E^B + w : B \subseteq C)$ edge coloring $c_E : E(G) \to C$, so we also give a vertex coloring $c_V : V(G) \to C$ that witnesses that c_E is balanced, and we combine these into a balanced full coloring $c = c_E \cup c_V$. Thus, we will color both edges and vertices of G such that every edge satisfies the endpoint rule.

If $G \in \Gamma + w$, then there exists W such that $(G, W) \in \Gamma \oplus w$. Thus $G - W \in \Gamma$ and so has a balanced $(k_E^B : B \subseteq C)$ full coloring. If we extend the coloring from $G - W$ to G by arbitrarily coloring W, and coloring edges incident with vertices arbitrarily subject to the endpoint rule, then the result follows, using Lemma 1.4(3).

Now suppose $G_1 \in \Gamma + w\Delta$ and $(G_2, W) \in \Gamma \oplus w$ and G is a clique-join of G_1 and G_2 on a clique Q with $V(Q) \subseteq W$. By induction there is a balanced $(k_E^B + w : B \subseteq C)$ full coloring of G_1 and there is a balanced $(k_E^B : B \subseteq C)$ full coloring of $G_2 - W$. Note that the graphs G_1 and $G_2 - W$ are vertex disjoint so we may use these two colorings to fully color $G_1 \cup (G_2 - W)$. This leaves the vertices in $W - V(Q)$, and the edges incident with these vertices, and the edges between $V(Q)$ and $V(G_2 - W)$, to be colored. We color these vertices and edges arbitrarily, subject to the endpoint rule, *except that* for any edge e with endpoints $u \in V(Q)$ and $v \in V(G_2) - V(Q)$, we set $c(e) = c(u)$.

For any $B \subseteq C$, there are no edges in $G|E(B)$ between $V(Q) - V(B)$ and $V(G_2) - V(Q)$, by the above choice of coloring for edges between $V(Q)$ and $V(G_2) - V(Q)$. Therefore $G|E(B)$ is a clique-join of $G_1|E(B)$ and $G_2|E(B)$, on the clique with vertex set $V(Q) \cap V(B)$. By Lemma 1.4(3), $tw(G_2|E(B)) \leqslant tw((G_2 - W)|E(B)) + w \leqslant k_E^B + w$. So by Lemma 1.4(2), $tw(G|E(B)) \leqslant k_E^B + w$, as required. $\square$

6. Proofs of main theorems

In this section, we prove the theorems stated in Section 1, as well as some other theorems. For a finite set of surfaces Θ let

$$2 - \chi(\Theta) = \max_{\Sigma \in \Theta}(2 - \chi(\Sigma)).$$

Theorem 6.1. *Let Θ be a finite set of surfaces, and let $r \geqslant 1$, $w \geqslant 0$ and $l \geqslant 2$ be integers. Then every graph in $(\Gamma_{\Theta,r}) + w\Delta$ has a balanced edge and a vertex l-coloring, such that any $j < l$ colors form a graph with tree-width at most*

$$(2 - \chi(\Theta))[2j(2r+1) + r + 2] + [3j(2r+1) + 2r + 3] + w.$$

Proof. This follows from Corollary 4.3 and Lemma 5.2. $\square$

Theorem 6.2. *Let Γ be a minor closed class of graphs other than the class of all graphs. Let $l \geqslant 2$ be an integer. Then there exist numbers α and β such that every graph in Γ has* *a balanced edge and a vertex l-coloring, such that any $j<l$ colors form a graph with tree-width at most $\alpha j+\beta$.*

Proof. This follows from Corollary 5.1 and Theorem 6.1. $\square$

The proofs of Theorems 1.1 and 1.2 follow immediately.

Finally, we need to prove Theorem 1.3. We first consider partitioning graphs of bounded tree-width. This topic was examined in [2], but we need a new formulation here.

A *k-tree* G is a graph with the following structure. Suppose $n>k$, let $V=V(G)=\{v_1,\dots,v_n\}$, and let $V_j=\{v_1,\dots,v_j\}$. For $k<j\leqslant n$, the neighborhood of v_j in $G|V_j$ is a k-clique. Observe that the largest clique in G has size $k+1$.

(Recall that, a subgraph of a k-tree is a partial k-tree and every (simple) graph with tree-width at most k is a partial k-tree.)

Let $C=\{1,\dots,k+1\}$ be a set of colors. We define *the canonical balanced full coloring* $c: V(G)\cup E(G)\to C$ *of* G, as follows. We color $c(v_i)=i$ for $i\leqslant k$, then we extend the vertex coloring uniquely, subject to being a *proper* vertex coloring. If edge e has endpoint v_i and v_j with $i<j$, then we color $c(e)=c(v_i)$.

Theorem 6.3. *Let $C=\{1,\dots,k+1\}$. If G has tree-width at most k, then G has a balanced full coloring $c: V(G)\cup E(G)\to C$, such that $G|V(B)$ is a $(|B|-1)$-tree and $G|E(B)$ is a $|B|$-tree for every $B\subseteq C$.*

Proof. Without loss of generality G is a k-tree. Let c be the canonical balanced full coloring. Let $V_j(B)=V_j\cap V(B)$. Let $E_j(B)=E_j\cap E(B)$. Observe that $G|(V_{k+1}(B))$ is a $|B|$-clique and for $k<j\leqslant n$, if $v_j\in V(B)$, then the neighborhood of v_j in $G|V_j$ is a $(|B|-1)$-clique. So $G|V(B)$ is a $(|B|-1)$-tree.

Now $G|E(B)$ is obtained from $G|V(B)$ by attaching each vertex in $V-V(B)$ to a $|B|$-clique. So $G|E(B)$ is a $|B|$-tree. $\square$

For a vertex coloring c, let π_c denote the corresponding vertex partition. The *meet*, $c_1\wedge c_2$, of two vertex colorings c_1 and c_2 is defined by $(c_1\wedge c_2)(v)=(c_1(v),c_2(v))$, that is, each vertex gets an ordered pair of colors. For a family of colorings $(c_i: i\in I)$, define $\bigwedge_{i\in I} c_i$ similarly. Make similar definitions for edge colorings and partitions. Clearly,

$$c=\bigwedge_{i\in I} c_i \Rightarrow \pi_c=\bigwedge_{i\in I}\pi_{c_i}.$$

Also,

$$|\pi_c|\leqslant \prod_{i\in I}|\pi_{c_i}|.$$

The following theorem sounds technical, but it is straightforward and very useful. For a set C and integer j let $\binom{C}{j}=\{B\subseteq C: |B|=j\}$.

Theorem 6.4. *The following statement holds for both edge and vertex colorings. Let Γ, Γ', and Γ_j, $j = 1, 2, \ldots$ be classes of graphs. Suppose, for all j, that Γ_j is closed under subgraphs (respectively, induced subgraphs) in the edge (respectively, vertex) coloring case. Let $j \geqslant 1$, $l \geqslant j$, $m \geqslant j$, be integers.*

(1) *Suppose that every graph in Γ can be l-colored so that every j colors form a graph in Γ'.*
(2) *Suppose that every graph in Γ' can be m-colored so that, for every $j' \leqslant j$, every j' colors form a graph in $\Gamma_{j'}$.*

Then there exists an integer i such that every graph in Γ can be i-colored so that, for every $j' \leqslant j$, every j' colors form a graph in $\Gamma_{j'}$.

Proof. We prove it in the edge case. (The vertex case is almost identical.) Let $G \in \Gamma$ and let $c : E(G) \to C$ be a coloring satisfying (1), where C is a set of colors of size l. For each $B \in \binom{C}{j}$, let $c_B : E(B) \to C_B$, be a coloring satisfying (2), where C_B is a set of colors of size m. Extend coloring C_B to all of $E(G)$ by assigning a single new color γ, say, to each edge in $E(G) - E(B)$, to yield $c'_B : E(G) \to C_B \cup \{\gamma\}$. Let $c' = \bigwedge_{B \in \binom{C}{j}} c'_B$.

Then c' is the desired coloring, and it uses at most $i \leqslant (m+1)^{\binom{l}{j}}$ colors. □

We may now prove Theorem 1.3. In fact it follows from the following stronger theorem, using Corollary 5.1.

Theorem 6.5. *Let Θ be a finite set of surfaces, and let $r \geqslant 1$, $w \geqslant 0$, and $j \geqslant 1$ be integers. There are integers i_V and i_E, such that every graph in $(\Gamma_{\Theta,r}) + w\Delta$ has a vertex partition into i_V graphs such that any $j' \leqslant j$ parts form a graph with tree-width at most $j' - 1$, and an edge partition into i_E graphs such that any $j' \leqslant j$ parts form a graph with tree-width at most j'.*

Proof. This follows from Theorems 6.1, 6.3 and 6.4. □

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